Resit Exam — Analysis (WBMA012-05)

Thursday 13 April 2023, 8.30h–10.30h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. Provide clear arguments for all your answers: only answering "yes", "no", or "42" is not sufficient. You may use all theorems and statements in the book, but you should clearly indicate which of them you are using.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (5 + 10 = 15 points)

Consider the set $A = \left\{ \dfrac{6x-5}{3x+4} \;:\; x > 0 \right\}$.

(a) Show that $u = 2$ is an upper bound for A.

(b) Prove that $\sup A = 2$.

Problem 2 (5 + 5 + 5 = 15 points)

Determine which of the following series converges or diverges. Motivate your answer!

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{9^k}{3^k + 6^k}$.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k}}{k^2 + 1}$.

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{p_k}$ where p_k is the k-th prime number (e.g. $p_1 = 2$ and $p_6 = 13$).

Problem 3 (3 + 3 + 9 = 15 points)

Let (a_n) be a convergent sequence and consider the set $A = \{a_n \;:\; n \in \mathbb{N}\}$.

(a) Give an example of a convergent sequence (a_n) for which A is compact.

(b) Give an example of a convergent sequence (a_n) for which A is *not* compact.

(c) Let $a = \lim a_n$. Show that $K = A \cup \{a\}$ is compact.

Please turn over for problems 4, 5, and 6!

Problem 4 (15 points)

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ x^2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Prove that f is differentiable at $x = 0$ and that $f'(0) = 0$.

Problem 5 (5 + 10 = 15 points)

Consider the following sequence of functions:

$$f_n(x) = \frac{n}{nx + 1}.$$

(a) Compute the pointwise limit for all $x \in (0, \infty)$.

(b) Let $a > 0$. Prove that the convergence is uniform on the interval $[a, \infty)$.

Problem 6 (3 + 12 = 15 points)

(a) Explain why the function $f(x) = 1/(1 + x)$ is integrable on $[0, 1]$ (e.g. by using a suitable theorem).

(b) Use the partition $P = \{k/n : k = 0, \ldots, n\}$ to prove the following inequality:

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \leq \ln(2) \quad \text{for all} \quad n \in \mathbb{N}.$$

End of test (90 points)

Solution of problem 1 (5 + 10 = 15 points)

(a) For $x > 0$ we have the following inequality:

$$\frac{6x-5}{3x+4} < \frac{6x+8}{3x+4} = \frac{2(3x+4)}{3x+4} = 2.$$

So for any $a \in A$ we have shown that $a < 2$ which implies that $u = 2$ is an upper bound for A.
(5 points)

(b) *Method 1.* Let u be any upper bound for A. Then for every $n \in \mathbb{N}$ we have

$$\frac{6n-5}{3n+4} \leq u.$$

(2 points)

By the Algebraic Limit Theorem it follows that

$$\lim \frac{6n-5}{3n+4} = \lim \frac{6-5/n}{3+4/n} = \frac{\lim(6-5/n)}{\lim(3+4/n)} = 2.$$

(3 points)

By the Order Limit Theorem it follows that $2 \leq u$.
(3 points)

So we have shown that any upper bound u of A satisfies $2 \leq u$. By the definition of supremum it follows that $\sup A = 2$.
(2 points)

Method 2. Let $\epsilon > 0$ be arbitrary. For $x > 0$ we have the following equivalent statements:

$$\begin{aligned} 2-\epsilon < \frac{6x-5}{3x+4} \quad &\Leftrightarrow \quad (2-\epsilon)(3x+4) < 6x-5 \\ &\Leftrightarrow \quad 8-4\epsilon < 3\epsilon x - 5 \\ &\Leftrightarrow \quad 13-4\epsilon < 3\epsilon x \\ &\Leftrightarrow \quad \frac{13-4\epsilon}{3\epsilon} < x. \end{aligned}$$

(5 points)

We conclude that for every $\epsilon > 0$ there exists an element $a \in A$ (namely, the element $a = (6x-5)/(3x+4)$ with $x > (13-4\epsilon)/3\epsilon$) such that $2-\epsilon < a$. This shows that any number $u < 2$ can is no longer an upper bound for A. Therefore, $\sup A = 2$ (we have used Lemma 1.3.8 here).
(5 points)

Solution of problem 2 (5 + 5 + 5 = 15 points)

(a) *Method 1.* For all $k \in \mathbb{N}$ we have the following inequality:

$$\frac{9^k}{3^k + 6^k} > \frac{9^k}{6^k + 6^k} = \frac{1}{2}\left(\frac{3}{2}\right)^k.$$

This shows that the terms of the given series do not converge to zero and therefore the series diverges.
(5 points)

Method 2. For all $k \in \mathbb{N}$ we have the following inequality:

$$\frac{9^k}{3^k + 6^k} > \frac{9^k}{6^k + 6^k} = \frac{1}{2}\left(\frac{3}{2}\right)^k.$$

For $r = 3/2$ the geometric series $\sum_{k=1}^{\infty} r^k$ diverges. By the Comparison Test the given series will diverge as well.
(5 points)

(b) For all $k \in \mathbb{N}$ we have the following inequality:

$$\frac{\sqrt{k}}{k^2 + 1} < \frac{\sqrt{k}}{k^2} = \frac{1}{k^{3/2}}.$$

In the lectures it was shown that any series of the form $\sum_{k=1}^{\infty} 1/k^p$ with $p > 1$ converges. Therefore, the Comparison Test implies that the given series converges as well.
(5 points)

(c) If p_k denotes k-th prime number, then the sequence $a_k = 1/p_k$ satisfies $0 < a_{k+1} < a_k$ for all $k \in \mathbb{N}$ and $\lim a_k = 0$. By the Alternating Series Test the given series converges.
(5 points)

Solution of problem 3 (3 + 3 + 9 = 15 points)

(a) Take, for example, the sequence (a_n) such that $a_n = 0$ for all $n \in \mathbb{N}$. Clearly, this sequence converges. We have that $A = \{0\}$ is a finite set and in the lectures it was proven that finite sets are compact.
(3 points)

Alternatives. Any constant sequence works. More generally, any eventually constant sequence works too.

(b) Take, for example, the sequence $a_n = 1/n$. Clearly, the sequence (a_n) converges. In this case, the set $A = \{1/n : n \in \mathbb{N}\}$ has $x = 0$ as a limit point which is *not* contained in A. We conclude that for this example the set A is not closed and hence not compact.
(3 points)

(c) Let O_λ, with $\lambda \in \Lambda$, be an open cover for the set K. Then for some $\lambda_0 \in \Lambda$ we have that $a \in O_{\lambda_0}$.
(1 point)

Since O_{λ_0} is open there exists $\epsilon > 0$ such that $V_\epsilon(a) \subseteq O_{\lambda_0}$.
(2 points)

Since $\lim a_n = a$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|a_n - a| < \epsilon$, or, equivalently, $a_n \in V_\epsilon(a)$.
(2 points)

Since the sets O_λ cover the set K, it follows that for the remaining indices $i = 1, 2, \ldots, N-1$ there exists a set O_{λ_i} such that $a_i \in O_{\lambda_i}$.
(2 points)

Finally, we conclude that $K \subseteq O_{\lambda_0} \cup O_{\lambda_1} \cup \cdots \cup O_{\lambda_{N-1}}$. This shows that any open cover for A has a finite subcover and thus that A is compact.
(2 points)

Solution of problem 4 (15 points)

For all $x \neq 0$ we have that

$$\frac{f(x) - f(0)}{x - 0} = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

(3 points)

This gives the following inequality for all $x \neq 0$:

$$\left| \frac{f(x) - f(0)}{x - 0} \right| \leq |x|$$

(3 points)

Let $\epsilon > 0$ be arbitrary and take $\delta = \epsilon$. Then we have the following implication:

$$0 < |x - 0| < \delta \quad \Rightarrow \quad \left| \frac{f(x) - f(0)}{x - 0} \right| \leq |x - 0| < \delta = \epsilon.$$

This shows that f is differentiable at $x = 0$ and that $f'(0) = 0$.
(9 points)

Solution of problem 5 (5 + 10 = 15 points)

(a) Let $x \in (0, \infty)$ be fixed. By the Algebraic Limit Theorem it follows that

$$\lim f_n(x) = \lim \frac{n}{nx+1} = \lim \frac{1}{x+1/n} = \frac{1}{\lim(x+1/n)} = \frac{1}{x+\lim 1/n} = \frac{1}{x}.$$

(5 points)

(b) We have

$$|f_n(x) - f(x)| = \left|\frac{n}{nx+1} - \frac{1}{x}\right| = \left|\frac{nx}{x(nx+1)} - \frac{nx+1}{x(nx+1)}\right| = \frac{1}{x(nx+1)}.$$

(3 points)

There are now at least two ways to finish the argument.

Method 1. Let $a > 0$ be fixed. If $x \in [a, \infty)$, then $x(nx+1) \geq a(na+1) > na^2$ so that

$$|f_n(x) - f(x)| < \frac{1}{na^2} \quad \forall\, x \in [a, \infty).$$

(2 points)

For $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $1/N < a^2\epsilon$. Hence,

$$n \geq N \quad \Rightarrow \quad |f_n(x) - f(x)| < \frac{1}{na^2} \leq \frac{1}{Na^2} < \epsilon \quad \forall\, x \in [a, \infty).$$

This shows that $f_n \to f$ uniformly on $[a, \infty)$.
(5 points)

Method 2. Let $a > 0$ be fixed. We have

$$\sup_{x\in[a,\infty)} |f_n(x) - f(x)| = \frac{1}{a(na+1)} < \frac{1}{na^2},$$

(5 points)

This implies that

$$\lim_{n\to\infty} \left(\sup_{x\in[a,\infty)} |f_n(x) - f(x)| \right) = 0.$$

This shows that $f_n \to f$ uniformly on $[a, \infty)$.
(5 points)

Solution of problem 6 (3 + 12 = 15 points)

(a) *Method 1.* The function is decreasing and in the lectures it has been shown that decreasing functions are integrable.
(3 points)

Method 2. The function is continuous and in the lectures it has been shown that continous functions are integrable.
(3 points)

(b) Since for $F(x) = \ln(1+x)$ we have $F'(x) = 1/(1+x)$, it follows by the Fundamental Theorem of Calculus that

$$\int_0^1 \frac{1}{1+x}\,dx = \ln(2) - \ln(1) = \ln(2).$$

(3 points)

Since f is decreasing it follows that

$$m_k := \inf\{f(x) \,:\, x \in [x_{k-1}, x_k]\} = f(x_k).$$

(3 points)

For the partition $P = \{k/n \,:\, k = 0, \ldots, n\}$ we thus get the following lower sum

$$\begin{aligned} L(f,P) &= \sum_{k=1}^{n} m_k(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} \frac{1}{1+k/n}\left(\frac{k}{n} - \frac{k-1}{n}\right) \\ &= \sum_{k=1}^{n} \frac{1}{n+k} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}. \end{aligned}$$

(5 points)

Finally, since $L(f,P) \leq \int_0^1 f$ for any partition P we obtain the desired inequality.
(1 point)